Tutorial 3

Microeconomics 3, Part 2

Question 1

Consider an economy with $I = 2$ consumers and $L = 2$ goods. There is a single firm whose only technology is free disposal. The consumers' utility functions over both goods are given by:

$$u_1(x_{11}, x_{21}) = 2\sqrt{x_{11}} + x_{21}$$
$$u_2(x_{12}, x_{22}) = x_{12} + bx_{22}$$

where $b \geq 0$ is a parameter. The consumption set for both consumers is $X_i = \mathbb{R}^2_+$. The aggregate endowment is given by $\bar{\boldsymbol{\omega}} = (1, 1)$.

(i) Consider the allocation $\boldsymbol{x}_1 = (0, 1)$ and $\boldsymbol{x}_2 = (1, 0)$. Is this allocation Pareto optimal? If the answer depends on the value of b, give different cases. If in a particular range of values of b it is not Pareto optimal, give an example allocation (that may depend on the value of b) that is Pareto improving.

(ii) Can the allocation in (i) arise in a competitive equilibrium for some initial endowments $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$? Again, consider different cases for the value of b.

(iii) Explain the relevance (or lack thereof) of the second welfare theorem in your answer to (ii).

Solution:

(i) We consider the $b = 0$ and $b > 0$ cases separately:

- Consider $b = 0$. In this case there is no way to make one consumer better off without making the other worse off. A (small) reduction in good 2 in exchange for an increase in good 1 for consumer 1 will make them better off, but this will make consumer 2 worse off. Similarly a (small) reduction in good 1 for consumer 2 in exchange for an increase in good 2 will make consumer 2 worse off. Consumer 2's utility is already maximized given the aggregate endowment. Thus it is Pareto optimal.

- Now consider $b > 0$. Both consumers obtain utility 1 under the current endowment. If we transfer $1/b$ units of good 2 to consumer 2 for each unit of good 1, we can keep them equally well off. But if we transfer δ units of good 1 from consumer 2 to consumer 1 and transfer $\frac{\delta}{b}$ of good 2 from consumer 1 to consumer 2 (to keep consumer 2 equally well off) we can increase consumer 1's utility, provided:

$$2\sqrt{\delta} + \left(1 - \frac{1}{b}\delta\right) > 1$$
$$2\sqrt{\delta} - \frac{1}{b}\delta > 0$$
$$2b > \sqrt{\delta}$$
$$4b^2 > \delta$$

  Therefore if $b > 0$, the proposed allocation is not Pareto optimal. An example of a Pareto improving allocation is to transfer $2b$ units of good 2 from consumer 1 to consumer 2 and to transfer $2b^2$ units of good 1 from consumer 2 to consumer 1. This gives both utility:

$$u_1 = 2\sqrt{2b^2} + (1 - 2b) = 2b\left(\sqrt{2} - 1\right) + 1 > 1$$
$$u_2 = 1 - 2b^2 + b \times 2b = 1$$

  Consumer 2 is equally well off, and consumer 1 is strictly better off, thus it is a Pareto improving allocation.

(ii) We consider the $b = 0$ and $b > 0$ cases separately:

- Consider $b = 0$. In this case there is no normalized price vector that can support this allocation in equilibrium. If $\frac{p_2}{p_1} > 0$, then consumer 1 will always want to demand a small amount of good 1 whereas consumer 2 will always want to keep $(1, 0)$. If $\frac{p_2}{p_1} = 0$, then consumer 1 will demand an infinite amount of good 2. If $\frac{p_2}{p_1} < 0$, then the firm can make unboundedly large profits by destroying a good.

- Now consider $b > 0$. We saw in (i) that in this case the allocation was not Pareto optimal. Because any competitive equilibrium in this economy must be Pareto optimal (because preferences are locally nonsatiated and the conditions of the first welfare theorem hold), the allocation cannot be part of a competitive equilibrium.

(iii) Only the $b = 0$ case is relevant, because for $b > 0$ the allocation was not Pareto optimal. Even though the allocation was Pareto optimal under $b = 0$, it cannot be part of a competitive equilibrium. This is not a violation of the second welfare theorem because not all conditions are satisfied. In particular, the theorem requires that $\boldsymbol{x}_i \gg \mathbf{0}$ for all i, which does not hold here. We are on the boundary.

Question 2

Consider an exchange economy with two commodities and two households. Household 1 owns six units of commodity 1 and two units of commodity 2, whereas household 2 owns ten units of commodity 1 and two units of commodity 2. Both households have the same preferences, represented by the utility function $u(x, y) = \sqrt{x} + 2\sqrt{y}$, where x is the consumption of commodity 1 and y is the consumption of commodity 2. Let $z(p)$ be the aggregate excess demand at price vector p.

(i) Derive $z(p)$ for any price vector p.

(ii) Show that z is bounded from below.

(iii) Show that z satisfies Walras' law.

(iv) Show that z satisfies homogeneity of degree zero.

(v) Show that z satisfies desirability, i.e., if the price of a commodity converges to zero, its excess demand goes to infinity.

(vi) Show that z satisfies continuity if all prices are positive.

(vii) Conclude that there exists a price vector p^* such that $z_1(p^*) = 0$ and $z_2(p^*) = 0$. Calculate this price vector (with sum of the prices equal to 1) and check that it is indeed a Walrasian equilibrium price vector.

(viii) Is z gross substitutable?

(ix) Conclude that this economy has only one Walrasian equilibrium (up to price normalization).

Solution:

Two consumers with utility function $u(x, y) = \sqrt{x} + 2\sqrt{y}$, consumer 1 owns (6,2), and consumer 2 owns (10,2).

(i) Aggregate excess demand function: $MRS(x) = \frac{1}{2}\frac{y^{1/2}}{x^{1/2}} = \frac{p_1}{p_2}$ yields $y = 4x\frac{p_1^2}{p_2^2}$. The budget constraint $p_1 x + p_2 y = M_i$ of consumer i with income $M_1 = 6p_1 + 2p_2$ for consumer 1 and income $M_2 = 10p_1 + 2p_2$ for consumer 2 then gives demand $x_i(p) = p_2 M_i/(p_1 p_2 + 4p_1^2)$ of good 1 and $y_i(p) = 4p_1 M_i/(4p_1 p_2 + p_2^2)$ of good 2 for consumer i. So, the aggregate excess demand function is $z(p) = (x_1(p) + x_2(p) - 16, y_1(p) + y_2(p) - 4) = (4\frac{p_2}{p_1} - 16, 16\frac{p_1}{p_2} - 4)$.

(ii) Bounded from below by any $-s \leq -16$: $z_1(p) > -16$ and $z_2(p) > -4$ for any $p \gg 0$.

(iii) Walras' law: $p \cdot z(p) = p_1 z_1(p) + p_2 z_2(p) = (4p_2 - 16p_1) + (16p_1 - 4p_2) = 0$ for any $p \gg 0$.

(iv) Homogeneity of degree zero: for every $\lambda > 0$ and $p \gg 0$ it holds that $z(\lambda p) = (4\frac{\lambda p_2}{\lambda p_1} - 16, 16\frac{\lambda p_1}{\lambda p_2} - 4) = (4\frac{p_2}{p_1} - 16, 16\frac{p_1}{p_2} - 4) = z(p)$.

(v) Desirability: If $p_1 \to 0$, then $z_1(p) = 4\frac{p_2}{p_1} - 16$ converges to infinity and if $p_2 \to 0$, then $z_2(p) = 16\frac{p_1}{p_2} - 4$ converges to infinity.

(vi) Continuity: Both $z_1(p) = 4\frac{p_2}{p_1} - 16$ and $z_2(p) = 16\frac{p_1}{p_2} - 4$ are continuous functions in p, $p \gg 0$.

(vii) Walrasian equilibrium: $z(p) = 0$ implies $p_1/p_2 = 1/4$. At this price ratio both consumers maximize their utility given their income and all markets clear. So, $(p_1^*, p_2^*) = (1/5, 4/5)$ yields a Walrasian equilibrium.

(viii) Gross substitutability: whenever $p'_l > p_l$ and $p'_k = p_k$, $k \neq l$, then $z_k(p') > z_k(p)$, $k \neq l$. Clearly this holds.

(ix) Uniqueness: From WARP (Prop. 17.F.2) or gross substitutability (Prop. 17.F.3) it follows that there is (at most) one Walrasian equilibrium.

Question 3

Consider a pure exchange economy with L goods and I consumers with preferences $\succeq_i$ over $X_i = \mathbb{R}^L_+$ that admit an aggregate demand function $\boldsymbol{z}(\boldsymbol{p})$ that is continuous over all $\boldsymbol{p} \in \mathbb{R}^L_+$ and satisfies:

- $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{z}(\alpha \boldsymbol{p})$ for all $\alpha > 0$ (homogeneity of degree zero).
- $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$ for all $\boldsymbol{p} \in \mathbb{R}^L_+$.

The distribution of the endowment vector satisfies $\boldsymbol{\omega}_i \gg \mathbf{0}$ for all i. There is a single firm whose production set is

$$Y_1 = \left\{ \boldsymbol{y}_1 \in \mathbb{R}^L : \boldsymbol{y_1} \leq \mathbf{0} \right\}$$

Each consumer i is entitled to a share $\theta_{i1} = \frac{1}{I}$ of the firm's profits.

Define the following function $f : \Delta \to \Delta$:

$$\{f_\ell(\boldsymbol{p})\}_{\ell=1}^L = \left\{ \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^K \max\{0, z_k(\boldsymbol{p})\}} \right\}_{\ell=1}^L$$

where:

$$\Delta = \left\{ \boldsymbol{p} \in \mathbb{R}^L_+ : \sum_{\ell=1}^L p_\ell = 1 \right\}$$

(i) [**2 points**] Show that Δ is convex.

(ii) [**2 points**] Show that Δ is compact.

(iii) [**2 points**] Show that if $\boldsymbol{p} \in \Delta$, then $f(\boldsymbol{p}) \in \Delta$.

(iv) [**4 points**] Prove that there exists a price vector $\boldsymbol{p}^\star \geq \mathbf{0}$, $\boldsymbol{p}^\star \neq \mathbf{0}$ that satisfies $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$.

(v) [**4 points**] If there is a price vector $\boldsymbol{p}^\star \geq \mathbf{0}$, $\boldsymbol{p}^\star \neq \mathbf{0}$ that satisfies $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$, is it a Walrasian equilibrium price vector?

Solution:

(i) Take $\boldsymbol{p}' \in \Delta$ and $\boldsymbol{p}'' \in \Delta$. We need to show that for any $\alpha \in [0,1]$ that $\boldsymbol{p}^\alpha = \alpha \boldsymbol{p}' + (1-\alpha)\boldsymbol{p}'' \in \Delta$. Each element of $\boldsymbol{p}^\alpha$ will be nonnegative by the nonnegativity of $\boldsymbol{p}'$, $\boldsymbol{p}''$ and $\alpha \in [0,1]$. Furthermore, $\sum_{\ell=1}^L p_\ell^\alpha = 1$ because:

$$\sum_{\ell=1}^L p_\ell^\alpha = \sum_{\ell=1}^L \left[\alpha p_\ell' + (1-\alpha) p_\ell''\right] = \alpha \sum_{\ell=1}^L p_\ell' + (1-\alpha)\sum_{\ell=1}^L p_\ell'' = \alpha + (1-\alpha) = 1$$

Therefore Δ is convex.

(ii) Δ is bounded because each p_ℓ must be nonnegative and sum to 1 (so no element of $\boldsymbol{p}$ can be unbounded). It is also closed. Take an arbitary sequence $\boldsymbol{p}^k$ converging to $\boldsymbol{p}$ where $\boldsymbol{p}^k \in \Delta$ for all k. We know that $p_\ell^k \geq 0\ \forall \ell$ and $\sum_{\ell=1}^L p_\ell^k = 1$ for all steps k in the sequence. Limits preserve weak inequalities and equalities so $\boldsymbol{p} \in \Delta$ as well. So Δ contains all of its limit points and is thus closed. Δ is closed and bounded so it is compact.

(iii) If $\boldsymbol{p} \in \Delta$, then:

$$\sum_{\ell=1}^L f_\ell(\boldsymbol{p}) = \sum_{\ell=1}^L \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^K \max\{0, z_k(\boldsymbol{p})\}} = \frac{\overbrace{\sum_{\ell=1}^L p_\ell}^{=1} + \sum_{\ell=1}^L \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{k=1}^K \max\{0, z_k(\boldsymbol{p})\}} = 1$$

And because $p_\ell \geq 0$ and $\max\{0, z_\ell(\boldsymbol{p})\} \geq 0$, and the denominator is strictly positive, we also have $f_\ell(\boldsymbol{p}) \geq 0$ for all ℓ. So $f(\boldsymbol{p}) \in \Delta$ as well.

(iv) The function $\boldsymbol{f}(\boldsymbol{p})$ is continuous (by the continuity of $\boldsymbol{z}(\boldsymbol{p})$ and the denominator being bounded away from zero) and maps vectors from a compact, convex and nonempty set to itself. Therefore by Brouwer it has a fixed point. Let $\boldsymbol{p}^\star$ be the fixed point. We just need to show that this fixed point price vector satisfies

$\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$. For each good we know that:

$$p_\ell^\star = \frac{p_\ell^\star + \max\{0, z_\ell(\boldsymbol{p}^\star)\}}{1 + \sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p}^\star)\}}$$

Multiplying across the denominator on the right hand side and canceling the $p_\ell^\star$ terms (just like in class):

$$p_\ell^\star \left(\sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p}^\star)\} \right) = \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

Multiplying both sides by $z_\ell(\boldsymbol{p}^\star)$ and summing over ℓ (just like in class):

$$\sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^\star) p_\ell^\star \left(\sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p}^\star)\} \right) = \sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

Looking at the left hand side:

$$\underbrace{\underbrace{\left(\sum_{k=1}^{K} \max\{0, z_k(\boldsymbol{p}^\star)\} \right)}_{\geq 0} \underbrace{\sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^\star) p_\ell^\star}_{\leq 0 \text{ from assumptions}}}_{\leq 0} = \sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\}$$

So we know that:

$$\sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^\star) \max\{0, z_\ell(\boldsymbol{p}^\star)\} \leq 0$$

Suppose there were any $z_\ell(\boldsymbol{p}^\star) > 0$. Then those elements of the sum over ℓ would be positive (squaring a positive number is positive). But no part of the sum can be negative: if $z_\ell(\boldsymbol{p}^\star)$ is negative, then that element of the sum is zero. So there can't be any positive $z_\ell(\boldsymbol{p}^\star)$ because otherwise the sum would never be nonpositive. Therefore $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$.

(v) Such a price vector is not necessarily a Walrasian equilibrium price vector. Walras' law was one of the necessary conditions for the existence proof we did in class, which is missing here. Walras' law says $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$ for all $\boldsymbol{p} \in \mathbb{R}_+^L$. Here we only have $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$ for all $\boldsymbol{p} \in \mathbb{R}_+^L$.

Because $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$, we know that $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0} \in Y_1$. Therefore producing $\boldsymbol{z}(\boldsymbol{p}^\star)$ is feasible for the firm. But because we don't have Walras' law, it is possible for $z_\ell(\boldsymbol{p}^\star) < 0$ while $p^\star_\ell > 0$ for some ℓ under the above assumptions. Therefore it is possible that $\boldsymbol{p}\cdot\boldsymbol{z}(\boldsymbol{p}) < 0$, which means the firm makes negative profits. The firm can always do nothing which gives zero profits: $\boldsymbol{y}_1 = \mathbf{0} \in Y_1$ and $\boldsymbol{p}\cdot\boldsymbol{y}_1 = \boldsymbol{p}\cdot\mathbf{0} = 0$ for all $\boldsymbol{p}$. Therefore producing $\boldsymbol{z}(\boldsymbol{p})$ may not be profit-maximizing for the firm given its technology, meaning it is not an equilibrium.